403

403

# THE MATHEMATICS STUDENT

*A Quarterly Dedicated to the Service of Students and Teachers of Mathematics in India*

Edited by
A. NARASINGA RAO, M.A., L.T.

OFFICE OF THE ...
19 OCT 1933
MADRAS

With the co-operation of

V. RAMASWAMI AIYAR, M.A.; G. A. SRINIVASAN, M.A., L.T.; N. SUNDARAM IYER, M.A.; C. N. SRINIVASIENGAR, D.Sc.; S. SIVASANKARANARAYANA PILLAI, M.Sc.; B. RAMAMURTI, M.A.; A. A. KRISHNASWAMI AYYANGAR, M.A.; S. PURUSHOTHAM, M.A.

*and others*

Published by
**The Indian Mathematical Society**

*in March, June, September and December*

Vol. I.] **June 1933.** [No. 2.

Printed by
SRINIVASA VARADACHARI & CO.
190, MOUNT ROAD, MADRAS.

JL
Bm 211, N33 MS
N33
190016

6.] [Single copy Rupee One and Annas Eight.

1623
23-9

# THE INDIAN MATHEMATICAL SOCIETY

was founded in 1907 for the "advancement of mathematical study and research in India." The Society has been formed on an All-India basis with its Head-Quarters centrally situated at Poona and the members of the Managing Committee elected so as to represent, as far as possible, the whole country.

Applications for membership should be sent to the Secretary, Prof. S. B. BELEKAR, M.A., College of Science, Nagpur, and have to be approved by the Managing Committee. The membership fee is Rs. 15 per annum, but the Committee has the power to reduce the subscription to Rs. 10 in the case of members whose salary does not exceed Rs. 250 per month. Members get free of all charges both the periodicals published by the Society, elect the office-bearers, and may borrow books from the Central Library at Poona.

## PUBLICATIONS.

The Society publishes two quarterlies:

*The Journal of the Indian Mathematical Society,*

which resumes publication from 1934, and will print original contributions of an advanced character, and

*The Mathematics Student,*

which is the official organ of the Society for all announcements and contains, besides questions and solutions, matter of a comparatively more elementary character, especially such as will be of interest to collegiate students and teachers. The annual subscription for either quarterly is Rs. 6 (Re. 1 and annas 8 per issue) and Rs. 9 for both together.

## LIBRARY.

The Society obtains both by exchange and by purchase, thirty leading mathematical periodicals of the world. These may be had on circulation by members and are later available for reference at the library at Poona. The library contains about 2,000 volumes of books and periodicals.

---


## RATES OF ADVERTISEMENTS

Advertisements relating to Mathematical books or Appliances will be accepted for publication in either quarterly at the following rates:

| | | *Inside page.* | *Inner or outer cover page.* |
|---|---|---|---|
| Full page | ... | Rs. 15 | Rs. 22 |
| Half ,, | ... | ,, 8 | ,, 12 |
| Quarter ,, | ... | ,, 4 | ,, 6 |

OFFICE OF THE REGISTRAR
19 OCT 1933
MADRAS

# THE MATHEMATICS STUDENT

Volume I.] JUNE 1933. [Number 2.

## NOTES ON THE THEORY OF NUMBERS: (I) PRIME NUMBERS.

By S. Chowla.

A natural number which has no positive integral divisors except itself and unity, is called a *prime* number. The primes upto 50 are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

It is usual to denote the $n$th prime by $p_n$, and we have $p_1 = 2$.

My object, in this note, is to give an account of a few striking results and problems in the theory of primes. The reader need have no previous acquaintance with the subject.

**Notation.**

We write

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_r^{\alpha_r}$$

where the $p$'s are distinct primes and $\alpha_m \geqslant 1$ $(1 \leqslant m \leqslant r)$;

$r = \varpi(n)$, the number of distinct prime factors of $n$;

$(a, b)$ is the greatest common divisor of $a$ and $b$;

$a/b$ if $a$ divides $b$;

$q \equiv a \pmod{c}$ if $c/(a - b)$;

$p$ denotes a typical prime;

$\pi(x)$ the number of primes not exceeding $x$;

$\pi(x; k, l)$ or $\pi_{k, l}(x)$ the number of primes $\equiv l \pmod{k}$ not exceeding $x$;

R. H. the Riemann hypothesis that no root of $\zeta(s) = 0$, where $= \sigma + it$, lies to the right of $\sigma = \frac{1}{2}$;

E. R. H. the 'extended Riemann hypothesis' that the zeroes of all Dirichlet's L-functions $L(s, \chi)$ lie on and to the left of $\sigma = \frac{1}{2}$;

$\phi(n)$ is the number of positive integers not exceeding $n$ and prime to it, $\phi(1) = 1$;

$f(x) = O(g(x))$ means that $\left| \frac{f(x)}{g(x)} \right| < K (< \infty)$ for all $x \geqslant x_0$;

$f(x) = o(g(x))$ means that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$;

$f(x) = \Omega_R(g(x))$ means that $\overline{\lim_{x \to \infty}} \frac{f(x)}{g(x)} > 0$;

$f(x) = \Omega_L(g(x))$ means that $\overline{\lim_{x \to \infty}} \left( -\frac{f(x)}{g(x)} \right) > 0$.

I use the logical signs $\to$ (if — then), & (and). Since the symbol $\to$ also used in the limit sense, the context will decide what is meant. Below a summation sign $\Sigma$, I first indicate the variables of summation and then the conditions of summation. Heavy type is used for reference to an author's books or papers.

In order not to frighten the unacquainted reader, I first give an introductory account of the results which find brief expression in the formulæ under the separate heads, I to IV below:

### The formulae I to IV.

(1) Expresses a long-sought-for result, now known as the Prime Number Theorem of Hadamard and de la Vallee Poussin (1896; Landau **1**). It is equivalent to

$$\lim_{n \to \infty} \frac{p_n}{n \log n} = 1.$$

Gauss noticed that the 'logarithmic integral'

$$\int_2^x \frac{du}{\log u}$$

150016

BM 211. N33 MS

is apparently a 'good' approximation for $\pi(x)$; this vague statement finds clear expression in Littlewood's (2) which is a sharper form of (1). While all available computational evidence seemed to show that

$$\pi(x) - \int_2^x \frac{du}{\log u} < 0$$

for large $x$, this was disproved by Littlewood (see Landau 2) as the $\Omega_R$ — half of (4) shows. Littlewood also disproved Tchebyscheff's conjecture that

$$\pi_{4,3}(x) - \pi_{4,1}(x) > 0$$

for all large $x$, as the $\Omega_L$ — half of (6) shows. Dirichlet proved that if $(k, l) = 1$, there are infinitely many primes $\equiv l \pmod{k}$. This 'deep' theorem is obviously included in (5).

While it seems likely that when $(k, l) = 1$ the least prime $\equiv l$ (mod. $k$) is $O(k^{1+\varepsilon})$ for every positive $\varepsilon$ and all large $k$, all that has been proved, and this on the assumption of infinitely many unproved hypotheses, is (8) [Titchmarsh 1].

Goldbach's celebrated (1742) conjecture is that every even natural number $\geqslant 6$ is a sum of two primes. The formulae in (II) embody the results of the best attempts in this direction. Viggo Brun proved (1, 10) that every large even number is a sum of two numbers each not having more than nine prime factors. This result was improved by Rademacher (1, 11) and recently by Estermann (1, 12).

Hardy and Littlewood proved (Landau 2) that every large odd number is a sum of three odd primes if the extended Riemann hypothesis is true. On the same assumptions they showed that 'almost all' even numbers are sums of two primes. Their method shows that every large number is a sum of not more than $c_1$ primes and not more than $c_2$ 1's where $c_1$ and $c_2$ are absolute constants, ***provided the extended Riemann hypothesis is true.*** That this result can be proved without any hypothesis is the astounding contribution of L. Schnirelmann (1; Landau 3), who used Viggo Brun's (1) method. Referring to Schnirelmann's paper, Landau (3) says: "Die Arbeit von Herrn L. Schni-

relmann enthält einen der grossten Fortschritte der Zahlentheorie die ich erlebt habe." Schnirelmann's theorem is embodied in (13).

(III) is devoted to the subject of the magnitude of the gaps between consecutive primes. For example, it has been conjectured that there are infinitely many groups of consecutive primes whose difference is 2. Such are

$$3, 5;\ 5, 7;\ 11, 13;\ \ldots\ \ldots;\ 101, 103;$$

see (21). The best that is known in this direction is Viggo Brun's theorem (1, 22).

Although it has not yet been shown that there is always a prime between the squares of any two consecutive natural numbers, Hoheisel has (1, 17) shown that there is always a prime for $n > n_0$ between $n$ and $n + n^c$ where $c > 1 - \frac{1}{33000}$ and $c < 1$. This result has recently been improved by Heilbronn (1, 18).

From the Prime Number Theorem it follows that for every positive $\delta$,

(i) $$\overline{\lim_{n \to \infty}} \frac{p_{n+1} - p_n}{\log n} > 1 - \delta,$$

(ii) $$\underline{\lim}_{n \to \infty} \frac{p_{n+1} - p_n}{\log n} < 1 + \delta.$$

Hardy and Littlewood (1, 19) have shown that if the extended Riemann hypothesis is true, then (ii) can be improved to

$$\underline{\lim}_{n \to \infty} \frac{p_{n+1} - p_n}{\log n} < \frac{2}{3} + \delta.$$

Erik Westzynthius has shown without any hypothesis (1, 20) that (i) can be improved to

$$\overline{\lim_{n \to \infty}} \frac{p_{n+1} - p_n}{\log n} = \infty.$$

In (IV) we find the conjecture that there are infinitely many primes $n^2 + 1$ and related results.

## I. The asymptotic distribution of primes.

(1) $$\lim_{x\to\infty} \frac{\pi(x)}{x/\log_e x} = 1.$$

(2) $$\pi(x) = \int_2^x \frac{du}{\log u} + O\left(xe^{-\sqrt{\log x \log\log\log x}}\right)$$

(3) $$\varepsilon > 0\ \&\ (\text{R. H.}) \rightarrow \pi(x) = \int_2^x \frac{du}{\log u} + O(x^{\frac{1}{2}+\varepsilon})$$

(4) $$\pi(x) - \int_2^x \frac{du}{\log u} = \Omega_{\substack{R\\L}}\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right)$$

(5) $$(k, l) = 1 \rightarrow \pi(x;\ k, l) = \frac{\{1 + o(1)\}\,x}{\phi(k)\log x}.$$

(6) $$\pi_{4,3}(x) - \pi_{4,1}(x) = \Omega_{\substack{R\\L}}\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right).$$

(7) $$\varepsilon > 0\ \&\ (k, l) = 1 \rightarrow \min_{\substack{\bar{\omega}(n) = 1\\ n \equiv l \pmod{k}\\ n > 1}} n = O(k^{1+\varepsilon}) \text{ for } k > k_0(\varepsilon).$$ (?)

(8) $$(\text{E. R. H.}) \rightarrow \min_{\substack{\bar{\omega}(n) = 1\\ n \equiv l \pmod{k}\\ n > 1}} n = O(k^{2+\varepsilon}) \text{ for } k > k_0(\varepsilon)$$

## II. Additive properties of primes.

(9) $$n \geqslant 3 \rightarrow \sum_{\substack{n_1, n_2\\ n_1 + n_2 = 2n\\ \bar{\omega}(n_1) = \bar{\omega}(n_2) = 1}} 1 > 0$$ (?)

(10) $$n \geqslant n_0 \rightarrow \sum_{\substack{n_1, n_2\\ n_1 + n_2 = 2n\\ \bar{\omega}(n_1),\ \bar{\omega}(n_2) \leqslant 9}} 1 > 0$$

(11) $$n \geqslant n_0 \rightarrow \sum_{\substack{n_1, n_2 \\ n_1+n_2=2n \\ \varpi(n_1), \varpi(n_2) \leqslant 7}} 1 > 0$$

(12) $$n \geqslant n_0 \rightarrow \sum_{\substack{n_1\, n_2 \\ n_1+n_2=2n \\ \varpi(n_1), \varpi(n_2) \leqslant 6}} 1 > 0$$

(13) $$n \geqslant n_0 \rightarrow \sum_{\substack{k, n_1, \ldots, n_k \\ n_1+\ldots+n_k=n \\ \varpi(n_1)=\ldots=\varpi(n_k)=1}} 1 > 0 \qquad (k < c)$$

(14) (E. R. H.) & $n \equiv 1 \pmod 2$, $n > n_0 \rightarrow \sum_{\substack{p_1, p_2, p_3 \\ p_1+p_2+p_3=n}} 1 > 0$

(15) (E. R. H.) & $n \equiv 0 \pmod 2$) $\rightarrow \sum_{\substack{n \\ n \leqslant x \\ n=p_1+p_2}} 1 = x/2 + o(x)$

## III. Gaps between consecutive primes.

(16) $$\sum_{\substack{p \\ n^2 < p < (n+1)^2}} 1 > 0 \qquad (?)$$

(17) $c > 1 - \frac{1}{33000} \rightarrow p_{n+1} - p_n = O(p_n^c)$

(18) $$\sum_{\substack{p \\ n^{250} < p < (n+1)^{250}}} 1 > 0$$

(19) (E. R. H.) & $\delta > 0 \rightarrow \varliminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log n} < \frac{2}{3} + \delta$.

(20) $$\overline{\lim_{n \to \infty}} \frac{p_{n+1} - p_n}{\log n} = \infty$$

(21) $$\sum_{\substack{n \\ \varpi(n) = \varpi(n+2) = 1}} 1 = \infty$$ (?)

(22) $$\sum_{\substack{n \\ \varpi(n), \varpi(n+2) \leqslant 9}} 1 = \infty.$$

IV. **Forms representing infinitely many primes.**

(23) $$\sum_{\substack{p \\ p = n^2 + 1}} 1 = \infty$$ (?)

(24) $$\sum_{\substack{n \\ n = x^2 + 1 \\ \varpi(n) \leqslant 7}} 1 = \infty$$

(25) $$\sum_{\substack{p \\ p = 2^n - 1}} 1 = \infty$$ (?)

(26) $$(a, b, c) = 1 \rightarrow \sum_{\substack{p \\ p = ax^2 + bxy + cy^2}} 1 = \infty.$$

---

*References.*

**Viggo Brun:**

1. Le crible d' Erathostene et le theoreme de Goldbach, *Videnskaps—selskapets skrifter*, I, *Mat. naturw Klasse.* No. 3, Kristiana, 1920, 36 pp.

**Estermann:**

1. *Jour. fur d. reine and angewandte Math.* (Crelle), 168, 1932, 106—116,

**Hardy and Littlewood :**

1. To appear in "Some problems of Partitio Numerorum, VII." The result is announced in P.N. VIII. *Proc. Lond. Math. Soc.*, 27, 1928, 518—242, (518).

**Landau :**

1. *Primzahlen* (Teubner, 1909).

2. *Vorlesungen uber Zahlentheorie* (1927).

3. Die Goldbachsche Vermutung und der Schnirelmannsche Satz *Gottinger Nachrichten, Math.-Phys. Klasse*, No. 10, 1930.

**Heilbronn :**

1. *Math.-Zeitschrift.*, 36, 1933, 394—423.

**Hoheisel :**

1. Primzahlprobleme in der Analysis, *Berliner Sitzungsberichte, Phys. Math. Klasse* , 1930, XXXIII.

**Rademacher :**

1. *Abhandlungen aus dem Mathematischen Seminar der Hamburgeschen Universität*, 3, 1923, 12—30.

**Schnirelmann :**

1. *Iswestija Donskowo Polytechnitscheskowo Instituta* (*Nowots cherkask*), 14, 1930, 3—28.

2. Über additive Eigenschaften von Zahlen, *Math. Annalen*, 107, 1933, 649—690.

**Titchmarsh :**

1. *Rendiconti del Circolo Matematico di Palermo* Tomo. LIV, 1930, 414—429.

**Erik Westzynthius :**

1. *Commentationes Phys.-Math. Soc. Sci. fenn*, 5, No. 25, 1—37 (1931).

# HISTORICAL NOTES IN FELIX KLEIN'S ELEMENTARY MATHEMATICS.

By G. A. Miller, *University of Illinois.*

The recent translation into English of volume I of F. Klein's ***Elementarmathematik vom hoheren Stundpunkte aus*** will dcubtless tend to extend the influence of this inspiring work among the teachers of mathematics in the countries where the English language is widely used. In the main, this influence will doubtless be very good since F. Klein had an unusually clear insight into the connection between various mathematical subjects and expressed himself clearly and forcibly. The great confidence inspired by his reputation and style tends to increase the difficulty involved in correcting the comparatively small number of errors which appear in this work. These errors are, however, especially dangerous in view of the fact that they appear where one would least expect to have to exercise caution.

In the present note, we shall confine our attention to the consideration of historical errors, and we begin with those relating to logarithms. On page 147 of the English translation, there appear the following statements: "It is very instructive to follow the train of thought of Napier and Bürgi. Both start from values of $x = b^y$ for integral values of $y$ and seek an arrangement whereby the numbers $x$ shall be as close together as possible. Their object was to find for every number $x$, as nearly as possible, a logarithm $y$."

The fact that neither Napier nor Bürgi started from the values of such an equation results directly from the following quotation from volume 2 (1933), page 211, of J. Tropfke's ***Geschichte der Elementar-Mathematik.*** "Beiden ist der Begriff der Grundzahl eines Logarithmen-systems gänzlich fremd. Sre beabsichtigen nur zwei Reihen, eine arithmetische und eine geometrische herzustellen, gliedweise zugeordnet, mit denen sie die von Stifel und Jacob angedeuteten Rechenerleichterungen ausführen wollen." This fact results also from the well-known property that the logarithm of 1 is not 0 in Napier's tables. (*Cf. Journal of the Indian Mathematical Soc.*, vol. 16, 1926, p. 209). It is unfortunate that the translators did not call attention to this

historical error in a foot-note, since it relates to a fundamental question in the development of logarithms, which has received much attention in recent years.

A modern historical note on the development of logarithms should not fail to emphasise the fact that both Napier and Bürgi were dealing with two simply isomorphic groups formed by their arithmetic series and geometric series respectively, which they placed in a (1.1) correspondence. In doing this, they failed, however, to make the identity of one of these groups correspond to the identity of the other, and hence their tables were soon superseded by more practical ones. It is misleading to refer to Napier's tables as the first tables of logarithms, as F. Klein does in the note in question, without pointing out that Napier's tables do not satisfy the fundamental condition that the logarithm of a product is the sum of the logarithms of its factors.

On page 80, it is stated that "the first beginnings of operating with letters were made also by the Hindus." It is well-known that the ancient Greeks were accustomed to represent surfaces, lines and points with one or more letters. Aristotle (384—322 B.C.) represented also other general quantities with letters and Plato called such letters mathematical numbers. Beginnings of operating with letters appear also in various parts of Euclid's *Elements*, and, according to T. L. Heath, "it should be added, as regards all the arithmetical books, that all numbers are represented in the diagrams as simple straight lines, whether they are linear, plane, solid, or any other kinds of numbers; thus the product of two or more factors is represented as a new straight line, not as a rectangle or a solid," *History of Greek Mathematics*, volume I, 1921, page 402.

On page 55, we read as follows: "Imaginary numbers are said to have been used first, incidentally, to be sure, by Cardan in 1545, in his solution of the cubic equation." Mathematical historians seem to agree that Cardan was the first to operate with complex numbers in his noted *Ars Magna*, 1545, but not with respect to the solution of the cubic equation. In fact, he selected the cubic equations which he solved in his *Ars Magna* and also in his later work, *De Regula Aliga*, 1570, in such a way that complex numbers did not appear in his solutions thereof. In his *Ars Magna*, he used complex numbers with respect to the

solution of a quadratic equation. While this might appear to be a somewhat minor point, it is important in view of the fact that complex numbers were first seriously developed in connection with the solution of the cubic equation, especially by R. Bombelli, and it is interesting to note that H. Cardan started in a different way.

While the preceding remarks aim to point out that the reader should be cautious as regards the use of the historical notes in the work under consideration, they do not aim to give the impression that these notes are worthless. The mathematical points of view brought out in connection with them are often very illuminating and exhibit that the writer thereof had a masterly grasp of the subject-matter. One does not find in these notes an emphasis on trivialities which have no bearing on mathematical ideas such as the listing of text-books which exhibit no mathematical advances, or references to unreasonable claims made for the attainments of ancient Chinese mathematicians. In fact, the writings of F. Klein are so eminently useful that it seems desirable to remove therefrom blemishes which might pass unnoticed in those of a less eminent mathematician.

# ON A PASTIME COMMON AMONG SOUTH INDIAN SCHOOL CHILDREN.

By S. Sivasankaranarayana Pillai, *Annamalainagar.*

**1.** To while away their time at odd intervals in school, sometimes two children engage themselves in the following easy and never-taxing game. Nine squares as in the following figure are drawn and by turns each marks a square. Only one square may be marked at a time. The first who is able to mark three squares in a straight line wins the game.

For example, let P and Q be the players, P the starter and $P_r$ and $Q_r$ be the squares marked respectively by P and Q in their *r*th turns. After P's third turn, let the state of the game be as in figure 1. Now, wherever Q may mark in his third turn, P will win the game in his fourth turn as he can get three markings either in the line $P_3P_2$ or in $P_3P_1$.

In order to save labour and time, the nine squares are indicated as in fig. 2, the understanding being that every open space in the figure also

| | | |
|---|---|---|
| $P_1$ | | $Q_2$ |
| | $Q_1$ | |
| $P_3$ | | $P_2$ |

Fig. 1.

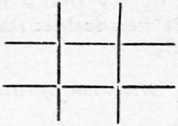


Fig. 2.

represents a square. Many may have noticed that, after some practice, the game loses its charm, because it always ends in a draw. The object of this note is to explain this fact and to consider some generalisations.

In the generalised form, we should have $n^2$ squares and the winner s he who first secures $t$ consecutive squares in a straight line. Since no oversight is presumed, unless one gets a two-way chance one cannot win the game. Since the starter is at advantage, in order to show that there will be always a draw for a particular $n$ and $t$, it is enough if we show that the opponent can, at each stage, prevent the starter from winning the game. When $t = n \geqslant 4$, we can prove the conclusion

in its general form. But the usual problem (*i.e.*, when $n = t = 3$) does not yield to general investigation. We have to prove the result by a method of enumeration.

**2.** First, we shall consider this case, $n = t = 3$. The total number of sets of three squares in a line is 8. Therefore, if Q, the opponent, exhausts 7 lines, P cannot get a two-way chance, and so it will end in a draw. So we need not proceed further. Let us name the squares like this :—

| | | |
|---|---|---|
| $B_1$ | $C_1$ | $B_2$ |
| $C_4$ | A | $C_2$ |
| $B_4$ | $C_3$ | $B_3$ |

By symmetry, it is enough if we consider the cases when P marks first A, $B_1$ or $C_1$. Let us consider those cases one by one.

*Case 1. Suppose P marks first the square A* :—

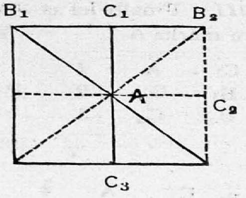
Let Q mark $B_1$. By that 3 lines are exhausted. Now by symmetry it is enough if we consider the cases when P marks in his second turn $B_3$, $B_2$, $C_1$ or $C_2$.

*Number of lines exhausted.*

| | P | Q | | |
|---|---|---|---|---|
| (1.1) | A | $B_1$ | 3 | |
| | $B_2$ | $B_4$ | 2 | P is compelled to mark $C_4$ |
| | $C_4$ | $C_2$ | 2 | |
| | | | 7 | |

| | | | | |
|---|---|---|---|---|
| (1.2) | A | $B_1$ | 3 | |
| | $B_3$ | $B_2$ | 2 | P is forced to mark C |
| | $C_1$ | $C_3$ | 2 | |
| | | | 7 | |

| | | | |
|---|---|---|---|
| (1.3) | A | $B_1$ | 3 |
| | $C_1$ | $C_3$ | 2 |




The dotted lines indicate the unexhausted lines. Only from $B_2$ and $C_2$ two unexhausted lines start. So, in order to get a two-way chance, P should mark in his third turn either $B_2$ or $C_2$. But, if P marks $B_2$, Q marks $B_4$ and gets a two-way chance. So P has to mark $C_2$. Thus we have

| | | | | |
|---|---|---|---|---|
| (1.3) | A | $B_1$ | 3 | |
| | $C_1$ | $C_3$ | 2 | |
| | $C_2$ | $C_4$ | 1 | P is forced to mark $B_4$ |
| | $B_4$ | $B_2$ | 1 | |
| | | | 7 | |

Lastly, we may have

| | | | | |
|---|---|---|---|---|
| (1.4) | A | B | 3 | |
| | $C_2$ | $C_4$ | 1 | P is forceed to mark $B_4$ |
| | $B_4$ | $B_2$ | 2 | ... ... $C_1$ |
| | $C_1$ | $C_3$ | 2 | |
| | | | 8 | |

*Case II.* Now let us consider the *case when P marks first the square* $B_1$. Q then marks A. Now, in order to get a two-way chance P should mark three B's with intervening squares vacant. Hence the following schemes follow :—

| | | | | |
|---|---|---|---|---|
| (2.1) | $B_1$ | A | 4 | |
| | $B_2$ | $C_1$ | 1 | P is forced to mark $C_3$ |
| | $C_3$ | $B_4$ | 2 | |
| | | | 7 | |

| | | | | |
|---|---|---|---|---|
| (2.2) | $B_1$ | A | 4 | |
| | $B_2$ | $C_1$ | 1 | P is forced to mark $C_3$ |
| | $C_3$ | $B_4$ | 2 | |
| | | | 7 | |

***Case III.*** Finally let as consider the ***case mhen P starts marking*** $C_1$. Q then marks A.

| | | | | |
|---|---|---|---|---|
| (3.1) | $C_1$ | A | 4 | |
| | $B_1$ | $B_2$ | 2 | P is forced to mark $B_4$ |
| | $B_4$ | $C_4$ | 1 | |
| | | | 7 | |

| | | | | |
|---|---|---|---|---|
| (3 2) | *a.* | $C_1$ | A | 4 |
| | | K | $B_1$ | 2 |
| | | $B_3$ | L | 1 |
| | | | | 7 |

Here K is any vacant square except $B_1$ and $B_2$. If $K \neq B_3$, P has to mark $B_3$. L is any vacant square.

| | | | | |
|---|---|---|---|---|
| *b.* | $C_1$ | A | 4 | |
| | $B_3$ | $B_2$ | 2 | P is forced to make $B_4$ |
| | $B_4$ | $C_3$ | 1 | |
| | | | 7 | |

Now we have exhausted all possibilities. So, if the game is carefully played it will end always in a draw.

**3.** Now we shall consider the general case when $t = n \geqslant 4$. In order to get a two-way chance, P should have marked at least $2n - 3$ ($= n - 2 + n - 2 + 1$) squares. Now the total number of lines is $2n + 2$. While P is marking $2n - 3$ squares, Q can, by suitably choosing two squares, prevent P from getting a two-way chance. For the other $2n - 5$ turns Q can mark wherever he chooses. But it is easily seen that he can mark $n - 1$ squares so that in each turn he can exhaust at least 2 lines, and in two squares (at the corners) he can exhaust 3 lines in each turn. If he has not exhausted all lines, in the other chances Q can mark squares so that in each turn he exhausts a line. So, the total number of lines that he can exhaust in $2n - 3$ turns is not less than $2(n - 3) + 2.3 + n - 2 = 3n - 2$. If $n \geqslant 4$, $2n - 5 \geqslant n - 1$; so Q can get $n - 1$ free choices. So, if $3n - 2 \geqslant 2n + 2$, *i.e.* if $n \geqslant 4$, the game will end in draw. So we see that the critical case is when $n = t = 3$.

Now let us consider the case when $t < n$. Let $r = n - t + 1$. It can be shown that the total number of lines with $t$ consecutive squares is $2rn + 2r^2$. It can be seen that Q can exhaust at least $2r$ lines in each turn by marking a square in the central portion with a margin of $r - 1$ squares all round. So he can get $n - 2r + 2$ such chances. In this case, to get a two-way chance P should have had at least $2t - 3$ chances. We shall assume that $2t - 5 \geqslant n - 2r + 2$. For each of the other chances, Q can exhaust at least one line. So the total number of lines that Q can exhaust when all the lines are not exhausted is

$$
\begin{aligned}
2r(n - 2r + 2) + 2t - 3 - (n - 2r + 2) \\
&= 2rn - 4r^2 + 2r + 2n + 2 - 2r - 3 - n + 2r - 2 \\
&= 2rn - 4r^2 + n + 2r - 3 \\
&\geqslant n + 2rn - 4r^2.
\end{aligned}
$$

So the game will end in a draw

if $\qquad n + 2rn - 4r^2 \geqslant 2rn + 2r^2$

*i.e.*, if $\qquad n \geqslant 6r^2$

*i.e.*, if $\qquad t \geqslant n + 1 - \sqrt{n/6}$.

Now we have proved if $t \geqslant n + 1 - \sqrt{n/6}$, then the game will end in a draw. But for lower values of $t$ also it may happen. Associated with every $n$ there is an $f(n)$ such that when $t \geqslant f(n)$, the game will end in a draw and when $t < f(n)$, the one who begins will win.

When $\qquad n = 3, \quad f(n) = 3$

$\qquad\qquad n = 4, \quad f(n) = 4$

$\qquad\qquad n = 5, \quad f(n) = 5$

$\qquad\qquad n = 6, \quad f(n) = 6$

and so on.

From the general result we get that $f(n) \leqslant n + 1 - \sqrt{n/6}$. But it seems to be difficult to find the exact value of $f(n)$ in the general case.

# ON THE LOCI OF THE CENTRES OF EQUILATERAL TRIANGLES INSCRIBED IN OR CIRCUMSCRIBED ABOUT OR SELF-POLAR TO A GIVEN CONIC AND OF REGULAR TETRAHEDRA SELF-POLAR TO A CONICOID.

N. DURAIRAJAN.

**1.** These loci are all obtained by a uniform method which is probably new. Individual methods have been given for obtaining the loci, as for example by Frank Morley and others.

Let the given conic be

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \qquad \ldots \qquad \ldots \quad (1)$$

The equation of the in-circle, circum-circle and polar-circle of an equilateral triangle may be taken to be

$$S_i \equiv (x - \alpha)^2 + (y - \beta)^2 - r^2 \;= 0 \qquad \ldots \quad (2)$$

$$S \;\equiv (x - \alpha)^2 + (y - \beta)^2 - 4r^2 = 0 \qquad \ldots \quad (3)$$

$$S_p \equiv (x - \alpha)^2 + (y - \beta)^2 + 2r^2 = 0 \qquad \ldots \quad (4)$$

The invariants of $S = 0$ and $S' = (x - \alpha)^2 + (y - \beta)^2 - R^2 = 0$ are easily seen to be

$$\Delta = -\frac{1}{a^2 b^2}; \qquad \Theta = \frac{1}{a^2 b^2}(\alpha^2 + \beta^2 - a^2 - b^2 - R^2)$$

$$\Theta' = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 - R^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right); \qquad \Delta' = -R^2.$$

**2.** *Locus of centres of equilateral triangles inscribed in S.*

Suppose the equilateral triangle of § 1 inscribed in the conic S. Then triangles can be inscribed in S, circumscribed about $S_i$ and self-polar

to $S_p$. Applying the usual conditions, we have

$$\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1-r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right] = 4(-r^2)\,\frac{\alpha^2+\beta^2-a^2-b^2-r^2}{a^2b^2}$$

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1+2r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=0,$$

Eliminating $r$, we get the locus of the centre to be the conic,

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)(9a^4+14a^2b^2+9b^4)=8(a^2+b^2)(x^2+y^2-a^2-b^2) \quad \ldots \quad (5)$$

or in a simpler form

$$x^2\,.\,\frac{(a^2+3b^2)^2}{a^2}+y^2\,.\,\frac{(b^2+3a^2)^2}{b^2}=(a^2-b^2)^2 \quad \ldots \quad (6)$$

The maximum and minimum equilateral triangles correspond to the turning values of $r$, and hence correspond to the values of $\alpha$, $\beta$ which make $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}$ a minimum or a maximum. Hence we deduce that the minimum and maximum equilateral triangles have for sides

$$\frac{4\sqrt{3}\,.\,ab^2}{a^2+3b^2} \quad \text{and} \quad \frac{4\sqrt{3}\,.\,a^2b}{3a^2+b^2}.$$

3. *Locus of centres of equilateral triangles circumscribed about* S.

Suppose the equilateral triangle of § 1 is circumscribed to S. Then triangles can be circumscribed about S, inscribed in $S_c$ and self-polar to $S_p$. Applying the usual conditions, we have

$$\left(\frac{\alpha^2+\beta^2-a^2-b^2-4r^2}{a^2b^2}\right)^2=\frac{-4}{a^2b^2}\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1-4r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right]$$

$$\alpha^2+\beta^2-a^2-b^2+2r^2=0.$$

Eliminating $r$, we obtain the locus of the centre in the form

$$(x^2+y^2-a^2-b^2)(9x^2+9y^2-a^2-b^2)+4a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0 \quad \ldots \quad (7)$$

(*Vide* Radford's *Problems*: p. 179)

It is easy to deduce that the sides of the minimum and maximum equilateral triangles are

$$\frac{2}{\sqrt{3}}\sqrt{2a^2 + 3b^2 \mp 2a\sqrt{a^2 + 3b^2}}.$$

The locus (7) is a bicircular quartic which is its own inverse with respect to a circle of radius $\sqrt{(a^2 - b^2)/3}$

It may be noted in passing that the locus of the vertices of circumscribed equilateral triangles is

$$3\,(x^2 + y^2 - a^2 - b^2)^2 = 4a^2b^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \quad \ldots \quad (8)$$

[The locus of the centre may also be found by eliminating R from the two known relations

$$(R^2 - OS^2)\,(R^2 - OH^2) = 4R^2 \,.\, b^2 *$$

and the result

$$OS.O.H. = 2R\,.\,d$$

when S, H are the foci, O the centre of the equilateral triangle, R the circum-radius and $d$ the distance of the centre of the conic from the centre of the nine-point circle.]

**4.** ***Locus of centres of the equilateral triangles self-polar to the conic S.***

Suppose triangles can be described so as to be self-polar to S, circumscribed about $S_i$ and inscribed in $S_c$. Applying the usual condition, we have

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 - r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 0$$

$$\alpha^2 + \beta^2 - a^2 - b^2 - 4r^2 = 0.$$

Eliminating $r$, we get the locus of the centre to be

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) - \frac{(a^2 + b^2)}{4a^2b^2}\,(x^2 + y^2 - a^2 - b^2) = 0 \ldots \quad (9)$$

---

* *Vide* V. Ramaswamy Aiyar: Some metrical relations connected with a pair of isogonal conjugates: *Jour. Ind. Math. Soc.*, Vol. III page 143, § 8, Cor. 1.

This may be written in the form

$$x^2(a^2 - 3b^2) - y^2(3a^2 - b^2) = (a^2 - b)^2 \quad \ldots \quad (10)$$

When S is an ellipse the locus is real when $a^2 > 3b^2$.

The envelope of the circles about the equilateral triangles is a bicircular quartic whose focal conic is the hyperbola (10) and whose fixed circle is the director circle of S.

The hyperbola possesses the interesting property that the angle between the two tangents from any point on it to the ellipse is half the angle between the two tangents to the director circle.

The side of the minimum self-polar triangle is given by

$$l = 2\sqrt{3} \, . \, \frac{b^2}{\sqrt{a^2 - 3b^2}}. \quad \ldots \quad (11)$$

5. *Regular tetrahedra specially related to a quadric.*

The same method can be applied to the problem of finding the locus of centres of regular tetrahedra self-polar to, or incident on, a given conicoid. But it is obvious that a locus—a curve—exists for self-polar tetrahedra and that in the other two cases, there can be no locus as such and that in these two cases it can be inferred that the centres should lie within a given surface.

Let the equation of the conicoid be

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad \ldots \quad (12)$$

The equations of the inscribed, circumscribed, polar sphere and the sphere touching the six edges of a regular tetrahedron can be written in the form

$$S_i = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - r^2 = 0 \quad \ldots \quad (13)$$

$$S_c = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - 9r^2 = 0 \quad \ldots \quad (14)$$

$$S_p = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 + 3r^2 = 0 \quad \ldots \quad (15)$$

$$S_e = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - 3r^2 = 0 \quad \ldots \quad (16)$$

The invariants of the conicoid S and the sphere

$$S' = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - R^2 = 0$$

are (*vide* Salmon and Rogers: ***Geometry of three dimensions***, p. 203).

$$\Delta = -\frac{1}{a^2b^2c^2}\,; \qquad \Delta' = -R^2$$

$$\Theta = \frac{1}{a^2b^2c^2}\left[\alpha^2 + \beta^2 + \gamma^2 - R^2 - a^2 - b^2 - c^2\right]$$

$$\phi = \frac{1}{b^2c^2}\left(\beta^2 + \gamma^2 - R^2\right) + \frac{1}{c^2a^2}\left(\gamma^2 + \alpha^2 - R^2\right)$$

$$+ \frac{1}{a^2b^2}\left(\alpha^2 + \beta^2 - R^2\right) - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$\Theta' = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 - R^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

## 6. *Regular Tetrahedra self-polar to given conicoid S.*

In this case tetrahedra can be described self-polar to S, inscribed in $S_e$, circumscribed to $S_i$, with its edges touching the sphere $S_e$. Applying the usual conditions, *vide* Salmon and Rogers, *loci cit.*

$$\alpha^2 + \beta^2 + \gamma^2 - 9r^2 - a^2 - b^2 - c^2 = 0\,;$$

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 - r^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 0\,;$$

$$\frac{1}{b^2c^2}\left(\beta^2 + \gamma^2 - 3r^2\right) + \frac{1}{c^2a^2}\left(\gamma^2 + \alpha^2 - 3r^2\right)$$

$$+ \frac{1}{a^2b^2}\left(\alpha^2 + \beta^2 - 3r^2\right) - \left(\frac{1}{a^2} + \frac{1}{b^2} - \cdot\frac{1}{c^2}\right) = 0.$$

Hence the centre lies on the curve of intersection of the conicoid given by P = Q = R where

$$P \equiv x^2 + y^2 + z^2 - a^2 - b^2 - c^2$$

$$Q \equiv 9\cdot\frac{a^2b^2c^2}{(b^2c^2 + c^2a^2 + a^2b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$R \equiv \frac{3}{(a^2 + b^2 + c^2)}\left[(x^2 + y^2 + z^2)(a^2 + b^2 + c^2)\right.$$

$$\left. - a^2x^2 - b^2y^2 - c^2z^2 - b^2c^2 - c^2a^2 - a^2b^2\right] \quad \ldots \quad (17)$$

The locus is a curve since there is a singly infinite system of tetrahedra; for a tetrahedron is defined by the twelve unknown

JL
BM211, N33 MS
120016

co-ordinates of its vertices which satisfy six conditions in virtue of the tetrahedra being self-polar to the conicoid, and five additional conditions due to the equality of the six edges.

7. (i) *Regular Tetrahedra inscribed in S.*

In this case, there are twelve unknown quantities which satisfy four conditions due to incidence on the conicoid, five conditions due to the equality of edges. The tetrahedra hence have three degrees of freedom and it is theoretically possible to describe a regular tetrahedron with its centre at any point in space. But in the case of the ellipsoid, it is obvious that the size of a real tetrahedron should lie within certain limits.

Applying the usual conditions, we get only one equation, *i. e.*, the vanishing of $\Theta'$

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 + 3r^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 0. \quad (18)$$

Hence if $r$ is given, we get a locus of centres, which is a coaxal conicoid. The locus is really the locus of the ortho-centres of orthogonal tetrahedra inscribed in the conicoid and having a given radius for their polar spheres. When the polar sphere is real, and the conicoid is an ellipsoid, the locus (18) lies entirely outside the ellipsoid. We obtain a locus even when the polar sphere is imaginary. When the polar sphere has a radius given by

$$\rho^2 = -\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)^{-1}$$

the locus shrinks to a point, which is the centre of the ellipsoid.

Obviously, more powerful methods are necessary to determine the bounding surface of centres of real regular tetrahedra.

(ii) *Regular tetrahedra circumscribed to a conicoid.*

The condition $\Theta = 0$ for S and $S_p$ is

$$\alpha^2 + \beta^2 + \gamma^2 + 3r^2 - a^2 - b^2 - c^2 = 0 \quad \ldots \quad (19)$$

For a given value of $r$, the locus is a sphere. A similar interpretaion, as in the last case, holds good.

## NOTES AND DISCUSSIONS.

*The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.*

### On isogonal point-pairs whose pedal circle cuts the nine points circle at a constant angle.

A pair of isogonal conjugates P, P′ with respect to a triangle determine a common pedal circle. By M'Cay's extension of Fuerbach's theorem

> the pedal circle of P and P′ touches the nine point circle if PP′ is collinear with S, the circum-centre of the triangle ... (1.1)

Since P and P′ coalesce at an in- or ex-centre, and hence fulfil the condition of collinearity with S in that position, Fuerbach's result that the in- and ex-circles of a triangle touch the nine point circle appears as a special case.

A criterion of a different character for the tangency of the nine point circle with a pedal circle was discovered by Youngman and independently by Bhimasena Rao* involving the position of P (or P′) only, namely :

> The pedal circle of P with respect to a triangle touches the nine point circle if the sum of the angles PAB, PBC, PCA is equal to one right angle. ... (1.2)

By a comparison of these two results, it follows that whenever PP′ cuts the circum-circle (or any circle concentric with it) at right angles, the common pedal circle of P and P′ touches the nine point circle.

This raises the question of a possible generalisation of (1.1), *i.e.*, of the existence of some simple relation between the angle of inter-

---

* *Jour. Ind Math. Soc.*, Vol. III, p 210, Question 329). *vide* also the Presidential Address by V. Ramaswamy Aiyar. *Report of the Fifth Conference of the Indian Math. Society*, pp. 12-13.

section of PP′ with some fixed circle concentric with S and the angle sum $\theta = P\hat{A}B + P\hat{B}C + P\hat{C}A$.

The interest in this enquiry is enhanced by the fact that Mr. V. Ramaswami Aiyar has obtained a generalisation of the other criterion (1.2) in proving that*

> the pedal circle of any point P cuts the nine point circle at the complement of the angle $\theta$. ... (1.3)

In order to obtain a generalisation of (1.1), the first step was naturally to study the envelope of PP′ when P takes all positions for which $\theta$ was a constant. This turns out to be a class sextic of genus one which in the special case when $\theta$ is a right angle breaks up into the four in-centres and the point S repeated twice. While a simple generalisation of (1.1) seems thus out of the question, the investigation itself is, perhaps, of sufficient interest to justify reproduction.

**2.** Consider in the plane of the complex variable $z = x + iy$, three points A, B, C represented by the complex numbers $a$, $b$, $c$ and let

$$f(z) = (z - a)(z - b)(z - c) \equiv P(x, y) + i\, Q(x, y). \quad \ldots \quad (2.1)$$

If P is any point whose affix is $z$, then $\angle PBC = \arg (z - b)/(c - b)$ so that the locus of P when $P\hat{A}B + P\hat{B}C + P\hat{C}A$ is constant is given by $\arg f(z) =$ const., that is, by

$$\frac{P(x, y)}{Q(x, y)} = c. \quad \ldots \quad (2.2)$$

Since P and Q are cubics in $x, y$ this gives a pencil of cubics which obviously passes through the points A, B, C since P can approach any of these parts in such a way as to give any desired value to $\theta$. This pencil of cubics is the conformal representation in the $z$-plane of the set of lines through $Z = 0$ in the transformation $Z = f(z)$. Writing (2.2) in the form

$$\frac{\{ f(z) + \overline{f(z)} \} / 2}{\{ f(z) - \overline{f(z)} \} / 2i} = c, \text{ that is, } \frac{\overline{f(z)}}{f(z)} = \frac{c + i}{c - i}$$

* *Journal of the Ind. Math. Soc.*, Vol. IX, p 130, (Question 863).

where $\overline{f(z)}$ is the conjugate of $f(z)$, we obtain the equation of the curves in the form

$$\frac{(x+iy-a)(x+iy-b)(x+iy-c)}{(x-iy-\bar{a})(x-iy-\bar{b})(x-iy-\bar{c})}=\frac{c+i}{c-i} \quad \dots \quad (2.3)$$

which again shows that the curve passes through the points $a, b, c$ since the equation is satisfied by $z = a, \bar{z} = \bar{a}$.

We may identify the Gauss plane with the Cartesian plane by associating with each real point $(x, y)$ the complex number $x + iy$, or if we prefer it, the pair of complex numbers $x + iy$ and $x - iy$. A point whose Cartesian co-ordinates $x, y$ are not both real may still be represented by two complex numbers $x + iy$ and $x - iy$ ; but they will not be conjugate imaginaries. With this mode of representation, it will be readily seen that the points A, B are represented by $(a, \bar{a})$ and $(b, \bar{b})$ while the intersections of the isotropic lines through A and B—the *cross pair* of A, B—are represented by $(a, \bar{b})$ and $(\bar{a}, b)$. The equation (2.3), is satisfied not only by the points $x + iy = a$, $x - iy = \bar{a}$, but also by the points $x + iy = a$, $x - iy = \bar{b}$, showing that the cubic (2.3) passes through the three points A, B, C, and the six cross pairs of these taken two at a time. These nine points constitute the base points of the cubic pencil.*

There is only one other point to note, namely, that if P traces out one cubic of the pencil (2.3) its isogonal conjugate P′ with respect to the triangle ABC traces another cubic of the same pencil. The angle sums $\Sigma$ PAB and $\Sigma$ P′AB are supplementary in this case. In particular

*when $\theta$ is a right angle, the cubic is self-isogonal and passes through the in- and ex-centres of the triangle.*

**3.** Let us denote by $C_\theta$ the locus of the point P for which $P\hat{A}B + P\hat{B}C + P\hat{C}A = \theta$ a constant angle $< \pi$. Then $C_{\pi-\theta}$ is the

---

* Gino Loria : *Spezielle algebraische und transzendente ebene Kurven.* I, pp. 448-449.

locus traced by the isogonal conjugate P′. We shall now determine the envelope of the line PP′.

If O is any arbitrary point in the plane, the locus of pairs of isogonal conjugates P, P′ collinear with O is a cubic (the isologue* of O) through A, B, C, the in- and ex-centres, O and O′. This cubic thus meets $C_\theta$ in six points in addition to A, B, and C, and the isogonal conjugate of these six points will lie on $C_{\pi-\theta}$. Since there are six lines PP′ through an arbitrary point O, the envelope of PP′ is a class sextic (besides the points A, B, C). This class sextic E being in (1 — 1) correspondence with the points of $C_\theta$ has the same genus, and is thus an elliptic curve.

We have seen that the cross pair of any two vertices AB lie on all the curves $C_\theta$. Since they are a pair of isogonal conjugates, on both $C_\theta$ and $C_{\pi-\theta}$ their join (which is the perpendicular bisector of AB) is a bitangent of the envelope Similarly the perpendicular bisectors of BC and CA are also bitangents of the envelope.

Again, either bisector of the angle A (say AI) meets $C_\theta$ and $C_{\pi-\theta}$ each in pairs of points, other than A. But as the bisector is a self-isogonal line, the pair on $C_\theta$ are the isogonal conjugates of the pair in which it cuts $C_{\pi-\theta}$. Hence it is a bitangent of the envelope E Hence we have proved that

*the envelope E of the lines PP′ is a class sextic having for bitangents the six joins of the points $I$, $I_1$, $I_2$, $I_3$, and the three lines through the circum-centre perpendicular to the sides.* ... (3.1)

We have thus obtained nine bitangents, the total number for a curve of unit deficiency.

4. It is worth watching how this sextic goes to pieces when $\theta = \pi/2$ in order to lead to the striking result (1.1). In this case $C_\theta$ becomes

* Hudson: *Cremona Transformations*, p. 45.

identical with $C_{\pi-\theta}$, and passes through the in- and ex-centres. As these points are self-isogonal, the line $PP'$ is indeterminate, and hence the class sextic breaks up into these four points and a residual conic. This latter must consist of the circum-centre S repeated twice, as it has three concurrent bitangents through S. Hence when $\theta = \pi/2$ the line $PP'$ always passes through S.

In fact when $\theta = \pi/2$, $C_\theta$ passes through the in- and ex-centres, the vertices A, B, C, and the points S, H. This is a self-isogonal curve and is the isologue of S.

It is thus seen that when $\theta$ is a right angle, we have no simple results analogous to (1.1), the only result along the same lines being (2.1).

*Annamalainagar.* A. NARASINGA RAO.

---

## GLEANINGS.

4\. ......... and working in this way, we suddenly get the given answer.

—*From an Intermediate answer book.*

5\. A Mathematical Proposition is a Hypothetical Proposition that has been established. Regarded as an enterprise, Mathematics is characterised by its aim, which is that of establishing Hypothetical Propositions. Regarded as a body of propositions, Mathematics consists of all and only such Hypothetical Propositions as have been established.

# ANNOUNCEMENTS AND NEWS.

*Readers are invited to contribute to the value of this section by sending suitable news items of interest.*

---

The following gentlemen have been admitted as members of the Society :

**Life Members.**

A. Narasinga Rao, Esq., M.A., L.T., Annamalainagar ; L. S. Vaidyanathan, Esq., M.A., F.I.A., Professor, Sydenham College, Bombay 1.

**Ordinary Members.**

Dr. Sudhanasukumar Banerji, D.Sc., Meteorologist, Ganesh Kind Road, Poona 5 ; G. R. Deo, Esq., M.A., Professor, Victoria College, Gwalior ; N. D. Doctor, Esq., B.A., Principal, New Bombay High School, Bombay 2 ; S. P. Kharas, Esq., B.A., Teacher, C Block, Gowalia Tank Buildings, Bombay 7 ; B. S. Madhava Rao Esq., M.Sc., Engineering College, Bangalore ; H. M. Mehta, Esq., B.A., Clerk, Agency Accounts Office, Bedigate, Rajkot ; Isaac Mathai, Esq., M.A., Research Scholar, Krishna Bhavan Hindu Colony, Dadar, Bombay ; Dr. S. H. Nanavathy, B.Sc., M.B.Bs., B.Hy., Ph.D., 41, Queen's Road, Bombay 2 ; M. R. Parthasarathy Ayyangar, Esq., M.Sc., Tutor, D 20, King's Circle, Matunga, Bombay 19 ; K. P. Patel, Esq., M.A. (Lond.), Asst. Professor, Robertson College, Jubbulpore ; A. Ranganatha Rao, Esq., Research Scholar, 8166, Saraswat Buildings, Bombay 19 ; N. A. Sastri, Esq., M.Sc (Lond.), Asst. Professor, College of Science, Nagpur, C.P. ; A. R. Sayad, Esq., B.Sc., B.T., Teacher, Raipur, Vagheswari Pole, Ahmedabad ; N. G. Shabde, Esq., M.Sc., Professor, City College, Nagpur ; S. M. Shah, Esq., M.A. (Lond.), F.R.S., Lecturer, Muslim University, Aligarh ; Srinivas Narayan Kawalgikar, Esq., M.Sc., Research Student, 508, Budhawarpet, Poona City.

The Academy of Sciences of the United Provinces of Agra and Oudh which was established to encourage research and co-ordinate scientific effort in the province has completed two years of valuable work and its anniversary was celebrated in January 1933 at Allahabad, the head-quarters of the Academy. A message from His Excellency Sir W. Malcolm Hailey, the Patron of the Academy, was read and the proceedings were presided over by the Hon. J. P. Srivastava,

---

A Scientific Proposition is a Categorical Proposition that has been established. Regarded as an enterprise, Science is characterised by its aim, which is that of establishing Categorical Propositions. Regarded as a body of propositions, Science consists of all and only such Categorical Propositions as have been established.

KEYSER : *The Pastures of Wonder.*

6. Sheer Mathematics is essentially and exclusively concerned with the world of the possible. It is *applicable* to the actual world because the world of the actual is part of the possible, else it could not be actual.

KEYSER : *The Pastures of Wonder.*

M.Sc., Minister of Education, U.P. The presidential address was delivered by the retiring President and Founder of the Academy, Dr. M. N. Saha, on "The Present Crisis in Dynamics." The office-bearers for 1933 are : ***President*** : Prof. K. N. Bahl, D.Phil. ; D.Sc., ***Vice-Presidents*** : Profs. M. N. Saha, D.Sc., F.R.S., B. Sahni, D.Sc., Sc.D. ; ***General Secretaries*** : Profs. P. S. MacMahon, M.Sc., F.I.C., and A. C. Banerji. M.A., M.Sc., I.E.S. The Academy publishes a ***Bulletin*** of which Volume I consisted of one, and Volume II of four numbers.

The Indian Science Congress will hold its 21st Annual meeting at Poona, on the 2nd, 3rd, 4th, 5th, 6th and 7th of January 1934. Dr. Meghnad Saha, D.Sc., F.R.S.. F.A.S.B., Professor of Physics, Allahabad University, will be President. Members who wish to communicate papers should forward them along with three copies of an abstract so as to reach the President of the Section concerned not later than the 1st October 1933. The President for the Mathematics and Physics Section is Dr. S. K. Mitra, D.Sc., Khaira Professor of Physics, Calcutta University, University College of Science, 92, Upper Circular Road, Calcutta.

Dr. P. L. Srivastava, Reader in Mathematics, Allahabad University, was elected a Fellow of the Academy of Sciences, U. P., in November 1932 "on account of eminence in scientific work." Prof. Srivastava's investigations have been on Integral Functions and Dirichlet's Series.

Dr. B. N. Prasad, Lecturer in Mathematics, Allahabad University, returned from Europe in July 1932 after obtaining the Ph.D. degree of the University of Liverpool and the degree of Docteur et Science (tres honourable) from Paris. Dr. Prasad's investigations relate to Fourier Series and their conjugate series, especially in relation to the problem of convergence and summability.

Dr. Syamadas Mukhopadhyaya M.A., Ph.D., Professor of Mathematics at the Calcutta University, who was awarded one of the Ghose Travelling Fellowships for 1932-33, left India for Europe in March 1933. for an advanced study of Topology and the methods of teaching modern mathematics at the present day in Europe. Dr. Mukhopadhyaya delivered some lectures at the invitation of the Paris University.

A foundation for the promotion of the international language "***Interlingua***," in which the late G. Peano was especially interested has been established in his honour under the auspices of the Journal. *School et Vita* : It will be called ***Fondo Peano pro interlingua***. Professor T. Levi Civita is a member of the committee in charge.

---

7. The number, the diameter and the circumference of islands, seas and mountains ; the enormous dimensions of the rows of habitations and halls belonging to the dwellers of this earthly world, of the interspace, of the world of the gods, of the world of light, and to the residents in hell, and miscellaneous statistics of various sorts—all these are realised by mathematics.

MAHAVIRA (850 A.D.)

8. Through and through the world is infected with quantity. To talk sense is to talk in quantities. You cannot evade quantity.

A. N. WHITEHEAD.

Prof. Albert Einstein has resigned his membership of the Prussian Academy of Sciences. He has accepted professorial appointments at the Universities of Madrid and Paris, and at the Institute for Advanced Study at Princeton. Among the other distinguished *savants*, who have joined the staff of the Institute of Advanced Study at Princeton are Prof. Herman Weyl of Gottingen and Prof. Oswald Veblen and Prof. Alexander of the Princeton University.

Dr. Solomon Lefschetz, Professor of Mathematics at Gottingen, has been appointed to the Henry Burchard Fine Chair of Mathematics to succeed Professor Oswald Veblen.

Dr. Sir C. V. Raman, D.Sc., LL.D., F.R.S., has been appointed Director of the Indian Institute of Science. Bangalore.

Dr. S. Chowla, M.A. (Panjab), PH D. (Cantab.), has been appointed Reader iu Mathematics in the Andhra University, Waltair, and joined duty on the 4th August.

The Syllabuses in Mathematics of the Bombay University have been revised so that the First-year syllabus includes two papers in Descriptive Mathematics as alternatives to the regular course in Algebra, Geometry and Trigonometry. For the Intermediate Examination, a paper in Statistics and Nomograms is an alternative to the paper in Analytical Geometry and Solid Geometry.

A new mathematical quarterly journal, *Scripta Mathematica* has been started by Yeshiva College, New York, under the editorship of J. Ginsburg, Professor of Mathematics, with an editorial board of prominent mathematicians. The journal is to be devoted "to the philosophy, history, and expository treatment of Mathematics" and will be entirely international in its scope. The subscription is three dollars per year.

Among new periodicals started in India within the last two years and dealing with Mathematics along with other subjects may be mentioned *Current Science* and the *Annamalai University Journal*. The former is a monthly published at Bangalore under the editorship of Prof. C. R. Narayana Rao, M.A. It is run on the same lines as *Nature* and publishes Papers, Letters to the Editor, Research Notes, Science News, Reviews, etc. It thus offers exceptional facilities for the quick communication of the latest results obtained. The price per volume of twelve issues is Rs. 6. The *Annamalai University Journal* is published by the Annamalai University under the editorship of Dr. B. V. Narayanaswami Naidu, Ph.D., Bar-at-law, Vol. I consisting of two issues, and the first issue of Vol. II are out. The annual subscription is Rs. 4. Other periodicals are: *Journal of the Osmania University* and *The Bombay University Journal*.

The Societa Reale di Napoli has agreed to present a copy of its *Rendiconti* in exchange for the publications of the Indian Mathematical Society.

Dr. R. T. A. Innes, formerly Director of the Union Observatory, Johannesburg, died on March 13th, aged 71.

Prof. Eugenio Bertini of the University of Pisa died on February 23rd, 1933, at the age of 86.

## REVIEWS.

**Intermediate Trigonometry**, by V. RANGANATHAN, B.A. (Hons.) and G. V. RAMACHANDRAN, B.A. (Hons.), (Wednesday Review Press, Trichinopoly). 1933. p. 225. Re. 1-12-0.

This little book is a successful attempt to write a small text-book, at a low price, to meet the needs of the Intermediate students of the Madras University. The topics have been divided into suitable groups in chapters of moderate length and examples sufficient in number and variety have been included in each. The authors have rightly devoted considerable attention to Trigonometrical graphs. That the formulae for compound angles are true for all values of the angles has received proper emphasis, and reference to Clark's *Mathematical and Physical Tables* gives a practical turn to the chapter on logarithms.

The following suggestions may be of help in case a second edition is called for: (1) the graphs of Trigonometrical functions may be placed one chapter earlier as they should not be made to depend on the properties of related angles, (2) the first treatment given for the derivation of general formulae for angles with a given sine, etc., is decidedly inferior to the one explained later on page 205 which deserves the first place, (3) the heading to section II, p. 94, is rather misleading, (4) some easy examples on heights and distances may be inserted earlier in the book, (5) the discussion of the ambiguous case in the solution of triangles may be taken up last as it is a comparatively difficult part of the theory.

The printing of the book admits of considerable improvement. Standard formulae, headings of sections, questions for solution, hints for their solution—everything is printed in one uniform type. The use of different types for different kinds of matter, and proper spacing between different sections and different chapters, would lessen the strain on the students' eyes and make the book more attractive.

N. R.

## SOLUTIONS TO QUESTIONS.

### Question 1325. (Second part)*

(B. B. BAGI) :—To prove that

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \dots\dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r = 0$$

has not more than one real root ($r < n$).

*Solution by C. N. Srinivasiengar.*

Let $f_r(x, n)$ denote the expression on the left.

The equation $f_2(x, n) \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 = 0$ has its roots imaginary for all values of $n > 2$.

Consider $f_3(x, n) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$.

Now $f'_3(x, n) = nf_2(x, n-1)$ which is positive if $n > 3$.

Hence $f_3(x, n)$ is monotonic increasing always, increasing from $-\infty$ to $+\infty$. Therefore $f_3(x, n) = 0$ has one and only one real root, if $n > 3$.

Let us now assume that $f_{r-1}(x, n-1)$ is monotonic increasing from $-\infty$ to $+\infty$, when $r$ is some even number $< n$. Let $\alpha$ denote its single real root.

Now $f'_r(x, n) = nf_{r-1}(x, n-1)$ which is positive if $x > \alpha$, and negative if $x < \alpha$. Hence, in the interval $(-\infty, \alpha)$ the function $f_r(x, n)$ is decreasing from $+\infty$ to $f_r(\alpha, n)$, and in the interval $(\alpha, +\infty)$ the function is increasing from $f(\alpha, n)$ to $+\infty$.

---

* The first part of this question was discussed in Vol. I, Number 1, p. 25, *et seq.*

The sign of $f_r(\alpha, n)$ which is thus the minimum value of $f_r(x, n)$ has to be ascertained.

Now, $$f_r(\alpha, n) = \sum_{k=0}^{r} {}_nC_k\ \alpha^k$$

while $$0 = f_{r-1}(\alpha, n-1) = \sum_{k=0}^{r-1} {}_{n-1}C_k\ \alpha^k.$$

Subtracting, we have

$$f_r(\alpha, n) = \alpha\ \{1 + (n-1)\alpha + \ldots\ldots {}_{n-1}C_{r-2}\ \alpha^{r-2} + \}\ {}_nC_r\ \alpha^r$$

$$= -\frac{(n-1)\ldots(n-r+1)}{(r-1)!}\alpha^r + \frac{n(n-1)\ldots(n-r+1)}{r!}\alpha^r$$

$$= \frac{(n-1)(n-2)\ldots(n-r+1)(n-r)}{r!}\alpha^r$$

which is positive when $n > r$.

Hence $f_r(x, n)$ is always positive, *i.e*, $f_r(x, n) = 0$ has no real root.

Next consider the equation $f_{r+1}(x, n) = 0$, ($r$ even).

Now, $f'_{r+1}(x, n) = n\,f_r(x, n-1)$ which is positive always if $r < n-1$. Hence $f_{r+1}(x, n)$ is monotonic increasing for all values of $x$, *i.e.*, $f_{r+1}(x, n) = 0$ has a single real root if $r + 1 < n$.

The proof is now complete by induction, since we have proved that $f_3(x, n) = 0$ has only one real root, and $f_2(x, n) = 0$ has none.

COROLLARY: *The equation*

$$F(x) \equiv \frac{1}{m!} + \frac{n-m}{(m+1)!}x + \frac{(n-m)(n-m-1)}{(m+2)!}x^2 + \ldots\ldots$$

$$\ldots + \frac{(n-m)\ldots(n-r+1)}{r!}x^{r-m} = 0$$

*has not more than one real root when* $m < r < n$.

This follows from Lemma III, by using

$$\frac{d^m}{dx^m}\left\{ x^m F(x) \right\} = 1 + \frac{n-m}{1!}x + \ldots + \frac{(n-m)\ldots(n-r+1)}{(r-m)!}x^{r-m}.$$

*Also solved by Hukam Chand and the Proposer.*

---

## Question 1566.

(A. A. KRISHNASWAMY AYYANGAR, M.A., L.T.):—Prove by direct algebra that the determinant of the sixth order

$$\begin{vmatrix} x-a & y-b & \xi\cos\theta & \xi\sin\theta & 0 & 0 \\ x-a' & y-b' & \eta\cos\phi & \eta\sin\phi & 0 & 0 \\ x-a'' & y-b'' & \zeta\cos\psi & \zeta\sin\psi & 0 & 0 \\ 0 & 0 & x-a & y-b & \xi\cos\theta & \xi\sin\theta \\ 0 & 0 & x-a' & y-b' & \eta\cos\phi & \eta\sin\phi \\ 0 & 0 & x-a'' & y-b'' & \zeta\cos\psi & \zeta\sin\psi \end{vmatrix} = 0$$

represents a pair of straight lines.

### *Solution by N. Sundaram Aiyar.*

By Laplace's expansion, the given determinant is equal to

$$\begin{vmatrix} x-a & y-b & \xi\cos\theta \\ x-a' & y-b' & \eta\cos\phi \\ x-a'' & y-b'' & \zeta\cos\psi \end{vmatrix} \times \begin{vmatrix} y-b & \xi\cos\theta & \xi\sin\theta \\ y-b' & \eta\cos\phi & \eta\sin\phi \\ y-b'' & \zeta\cos\psi & \zeta\sin\psi \end{vmatrix}$$

$$-\begin{vmatrix} x-a & y-b & \xi\sin\theta \\ x-a' & y-b' & \eta\sin\phi \\ x-a'' & y-b'' & \zeta\sin\psi \end{vmatrix} \begin{vmatrix} x-a & \xi\cos\theta & \xi\sin\theta \\ x-a' & \eta\cos\phi & \eta\sin\phi \\ x-a'' & \zeta\cos\psi & \zeta\sin\psi \end{vmatrix}$$

$= \Delta_1\Delta_2 - \Delta_3\Delta_4$ (say).

Also each of the determinants equated to zero, represents a straight line. If $\Delta_4$ be written in the form

$$A(x-a) + B(x-a') + C(x-a'') = 0, \qquad \ldots \quad (1)$$

then $\Delta_2$ is

$$A(y-b) + B(y-b') + C(y-b'') = 0. \qquad \ldots \quad (2)$$

Also clearly

$$A(\xi\cos\theta) + B(\eta\cos\phi) + C(\zeta\cos\psi) = 0, \qquad \ldots \quad (3)$$

and

$$A(\xi\sin\theta) + B(\eta\sin\phi) + C(\zeta\ \text{in}\ \psi) = 0. \qquad \ldots \quad (4)$$

Hence, unless A, B and C are simultaneously zero, in which case, the given determinant is identically zero, we have from (1), (2) and (3)

$\Delta_1 = 0$ and from (1), (2) and (4), $\Delta_3 = 0$, as consequences of the simultaneous relations $\Delta_2 = 0$ and $\Delta_4 = 0$.

Hence the four straight lines $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$ and $\Delta_4 = 0$ are concurrent. Hence $\Delta_1\Delta_2 - \Delta_3\Delta_4 = 0$ represents a pair of straight lines through this point of concurrence.

***Also solved by S. P. Ranganathachar.***

---

## Question 1568.

(B. RAMAMURTI) :—ABC is a triangle inscribed in a conic. A′,B′,C′ are respectively the harmonic conjugates of A with respect to B and C, of B with respect to C and A, of C with respect to A and B. P is any point on the conic and $P_1, P_2, P_3$ are respectively the harmonic conjugates of P with respect to (AA′), (BB′), and (CC′). Show that $(AP_1)$, $(BP_2)$, $(CP_3)$ belong to an involution of which the double points are the intersections with the conic of the polar axis of P with respect to ABC. Show also that $(A'P_1)$, $(B'P_2)$ and $(C'P_3)$ belong to an involution and that the Hessian pair of ABC is the common pair of the two involutions.

*Solution by the Proposer.*

Expressing the coordinates of any point on a conic in terms of a parameter $t$, the triad ABC can be reduced to the canonical form $t^3 + 1 = 0$. Then the triad A′B′C′ is given by $t^3 - 1 = 0$. The common Hessian pair is given by $0 . t^2 + t = 0$. Now the polar dyad of P (T) with respect to ABC is given by $T t^2 + 1 = 0$.

Hence the polar dyad is the pair of double points of the involution determined by A (—1), $P_1$ (1/T), $H_1$, $H_2$. Similarly, $BP_2$, $CP_3$ also belong to the involution. But the polar dyad is given by the intersection with the conic of the polar axis of P with respect to ABC. Hence the first part.

Similarly $(A'P_1)$, $(B'P_2)$ $(C'P_3)$ belong to an involution the double points being the polar dyad of P with respect to A′B′C′ and the Hessian points belong to both of these involutions.

---

### Question 1576.

(A. RANGANATHA RAO) :—O is any point on the nine point circle of a triangle ABC and D, E, F are the middle points of BC, CA, AB respectively. Prove that the lines through the ortho-centre of ABC parallel to OD, OE, OF meet BC, CA, AB respectively in three points which are collinear.

*Solution by the Proposer.*

It is known that a rectangular hyperbola circumscribing a triangle has its centre on the nine point circle. Take the rectangular hyperbola which has O for its centre. Let this figure be contained in a plane $p$. Project the figure on to a plane $p'$ so that the two points in which the line at infinity on $p$ cuts the hyperbola are projected into the circular points on $p'$. The line at infinity in $p$ projects into the line at infinity in $p'$. Parallel lines project into parallel lines. Middle points of segments of lines project into the middle points of projected segments. Since O projects into the circum-centre of the projected triangle, we must show on $p'$ that lines drawn from a point on the circum-circle parallel to the perpendicular bisectors of the sides of the triangle meet the corresponding sides in three points in a line. This is in fact the pedal line.

*Solved analytically by K. V. Krishna Sastry and by the application of Menelaus' Theorem, by S. Narayanan and Venkatasubramaniam.*

---

### Question 1643

(A. A. KRISHNASWAMI AIYANGAR) :—O is a fixed point. P, Q are a point-pair belonging to a given involution on a line AB. Prove that the perpendiculars from P, Q on OQ, OP envelope a parabola.

*Solution by V. Ramaswamy Aiyar.*

The circles described on PQ as diameter, where P, Q are a variable point-pair of the involution, form a co-axal system. One circle of the system will pass through O. Let $P_0$, $Q_0$ be the particular point-pair determining this circle. We have then a pair $P_0$, $Q_0$ belonging to the involution such that $OP_0$, $OQ_0$ are perpendicular to one one another.

Let M be the centre of the involution, that is, the point corresponding to that at infinity on AB. At M erect a perpendicular $m$ to AB,

Let $\Gamma$ denote the parabola touching the sides of $OP_0Q_0$ as well as the line $m$.

Let the tangents drawn from P, Q to $\Gamma$ intersect in Y. The locus of Y as the point-pair P, Q varies is a straight-line $l$, by the (simple) lemma noted below. Considering $P_0$, $Q_0$ as one case of the pair P, Q we see that O lies on the line $l$. Considering M and the point at infinity on AB as another case of the pair P, Q we see that $l$ passes through the point at infinity on $m$. Hence the line $l$ is the perpendicular drawn from O on AB; and Y lies on this perpendicular.

Now consider the ortho-centre of the triangle YPQ. It must lie on the perpendicular from Y on PQ, that is, the line $l$. Since YPQ circumscribes the parabola $\Gamma$, the ortho-centre must lie on the directrix. The directrix is evidently OM, since $OP_0$, $OQ_0$ are one pair of perpendicular tangents to $\Gamma$ and AB and $m$ (intersecting at M) are another. The ortho-centre of YPQ is accordingly the intersection of $l$ with OM, ***that is, it is the point*** O.

Consequently, the ortho-centre of OPQ is Y; and therefore PY, QY (the tangents to the parabola from P, Q) are perpendicular to OQ, OP respectively. Hence, conversely, the perpendiculars drawn from P, Q on OQ, OP envelope a parabola, namely, $\Gamma$.

The lemma referred to above can be stated thus:

If X be a conic, and $\lambda$ a line not touching it, the pairs of tangents drawn to X from a variable point of $\lambda$ cut any fixed tangent $t$ to the conic in a pair of points forming an involution.

***Conversely***, if from points P, Q, forming an involution on a line $t$, tangents be drawn to a conic X touching the line, the locus of their intersection is a straight line $\lambda$.

*Remarks by* ***A. Narasinga Rao.***

There is one obvious exceptional case when the envelope is ***not a parabola*** but the point O, namely, when the involution pencil (OP, OQ) is orthogonal. In the other cases it is a parabola.

Using Mr. Ramaswami Aiyar's notation, let $PP'$ denote the line through P perpendicular to OQ, the point $P'$ being at infinity. $OP'$ is then parallel to $PP'$ and hence perpendicular to OQ. We thus have two ranges of points (P) and (P′) on the line AB and on the line at infinity which are projective, since

$$(P) \equiv (Q) \equiv O(Q) \equiv O(P') \equiv (P'),$$

the symbol ≡ indicating that the ranges and pencils are projective (equicross). Hence either the joins of corresponding points PP′ form a pencil (exceptional case mentioned above), or they envelop a conic* touching AB and the line at infinity. That this parabola also touches *m* may be seen by taking P at M.

More generally, let lines be drawn through P making any constant angle $\alpha$ with OQ.

These lines may be obtained by rotating the lines PP′ about P in the same sense through a constant angle $\beta = \alpha - 90°$. If the line PP′, after rotation, cuts the line at infinity in P″, the ranges (P′) and (P″) are projective, so that $(P) \equiv (P') \equiv (P'')$.

Hence in this more general case also the envelope of PP″ is a point or a conic $\Gamma_\alpha$. In the latter case, it must be a parabola touching AB as may be seen by taking Q at M, and OQ making an angle $-\alpha$ with AB respectively.

Let I, J be the circular points at infinity and let OI, OJ cut AB in $i$ and $j$. Denote by $i'$ and $j'$ the conjugates of $i$ and $j$ in the involution. When Q is at $i$ or $j$, the line PP′ takes the positions I$i'$ or J$j'$. Hence the intersection F of I$i'$ and J$j'$ is the focus of $\Gamma$. To find the focus of $\Gamma_\alpha$, we notice firstly that every tangent to $\Gamma$ turned about its intersection with AB through the constant angle $\alpha - 90°$ becomes a tangent to $\Gamma_\alpha$, and secondly that the analytical transformation corresponding to the rotation leaves the lines I$i'$ and J$j'$ invariant. Hence the focus of $\Gamma$ is identical with that of $\Gamma_\alpha$.

By varying $\alpha$, we thus obtain a system of parabolas $\Gamma_\alpha$ having the same focus F and touching the line AB, that is, a four-line system of conics. The only *real* degenerate member of this system consists of two points—F and the point at infinity on AB. Hence the exceptional case occurs for some value of $\alpha$, if not for $\alpha = 90°$.

***Analytical Solution by V. Ranganathan, B.A. (Hons.)***

---

* This is the orthodox text-book statement that the lines PP′ are concurrent, if the intersection T is a self-corresponding point of the two ranges, and in other cases envelop a conic. When P and P′ coincide at T, the line PP′ may be any line through T, so that it would be more proper to say that the envelope consists of two points one of which is T. Otherwise, it sounds absurd to speak of an envelope, usually a conic, degenerating under special circumstances into a single point.

## QUESTIONS FOR SOLUTION.

*Proposers of Questions are requested to send their own Solutions along with their Questions.*

**1644.** (C. N. SRINIVASIENGAR) :—If a plane curve possesses a ramphoid cusp, the two branches of the curve at the cusp, the first polar of any point not on the cuspidal tangent, and the three partial branches of the Hessian at the cusp, all have the same curvature at the point.

**1645.** (A. NARASINGA RAO) :—On each tangent to a class $n$ – ic C, are taken the pair of points P, P′ which are isogonal conjugates with respect to a triangle. Discuss the relation of C to the triangle necessary and sufficient to ensure that the locus of P and of P′ be algebraically distinct curves.

**1646.** (A. A. KRISHNASWAMI AYYANGAR) :—Show that the equation of the axis of the parabola S = 0 in rectangular cartesian co-ordinates can be put in either of the forms

$$\frac{\partial}{\partial x}\left(S_x^2 + S_y^2\right) = 0; \quad \frac{\partial}{\partial x}\left(S_x^2 + S_y^2\right) = 0$$

where $S_x$ and $S_y$ stand for $\frac{\partial S}{\partial x}$ and $\frac{\partial S}{\partial y}$ respectively.

Explain the geometrical significance of the above forms.

**1647.** (A. A. KRISHNASWAMY AYYANGAR) :—If confocal conics reciprocate into co-axal circles with respect to a conic, then this conic is either a circle or a rectangular hyperbola with the same centre.

(The existence of the R. H. is not usually pointed out in text-books).

**1648.** (V. RAMASWAMI AIYAR) :—Two triangles ABC, A′B′C′ are in perspective. If A′B′C′ be moved rigidly and without rotation, so as to be constantly in perspective with ABC, show that the locus of any point P of its plane (of its *body*, if you please) is, in general, a conic.

In what case or cases, will the locus of P be a line, apart from the obvious case where the centre of perspective O is at infinity when P can move straight towards O ?

**1649.** (V. RAMASWAMI AIYAR) :—Given two tetrahedra ABCD, A′B′C′D′ in perspective, investigate if one of them can be generally moved, rigidly and without rotation, so as to be always in perspective with the other.

(The proposer seeks enlightenment on this question).

**1650.** (V. RAMASWAMI AIYAR) :—Two triangles ABC and A′B′C′ are in perspective. If any conics whatever circumscribe the triangles and pass through the centre of perspective, O, show that they cut again in three points which form a triangle self-conjugate with respect to a fixed conic.

**1651.** (V. RAMASWAMI AIYAR) :—(*a*) Two triangles ABC, A′B′C′ are polar reciprocals with respect to a parabola. If from any point O vectors OL, OM, ON be set up equal to the vectors AA′, BB′, CC′, show that L, M, N are collinear

(*b*) Show that a like theorem holds for two tetrahedra ABCD, A′B′C′D which are polar reciprocals with respect to a paraboloid.

**1652.** (A. A. KRISHNASWAMI AYYANGAR) :—Tangents at two points P, Q of a conic meet in T. Show that the circle passing through T and touching PQ at P or Q cut orthogonally the director circle of the conic. Hence deduce Gaskin's theorem of which the above theorem is a special case : "The circum-circle of a triangle self-conjugate with respect to a conic cut the director circle orthogonally."

(The moral from the above question is that it is sometimes as easy to deduce the general from the particular as to deduce the particular from the general).

**1653.** (K. J. SANJANA and K. F. VAKIL) :—In the triangle ABC each of the angles at B and C is double of the angle at A ; CM is drawn perpendicular to AB, AM and MB are bisected at F and G respectively and CF and CG are joined.

Prove that

(1) $\sin FCG : \sin ACB = 8 : 11$ ;

(2) rect. $AF . BG$ : rect. $CF . CG = 1 : 11$ ;

and (3) in-radius of $\triangle CFG$ : in-radius of $\triangle CAB = 5 + \sqrt{5} : 10$.

190016

JL

Bm21';N53.MS
N33

## List of Journals and Books received in the Library from December, 1932 to July, 1933.

1 Abhandlungen aus dem Mathematichen Seminar, Hamburg, **9**, 2, 3, 4.
2 Acta Mathematica, **60**, 3, 4.
3 American Journal of Mathematics, **55**, 1, 2 (2 copies).
4 American Mathematical Monthly, **39**, 9, 10, **40**, 1 to 6 (3 copies each)
5 Annales de L'Ecole Normale Superieure, Nov., 1932 to June, 1933.
6 Annals of Mathematics, **34**, 1, 2 (2 copies each).
7 Astrophysical Journal, **77**, 1 to 5.
8 Boletin del Seminario Matematico, **3**, 8 to 12.
9 Bulletin of the American Mathematical Society, **38**, 10, 11, 12, **39**, 1 to 6.
10 Bulletin des Sciences Mathematiques, **56**, June, Oct., Nov., Dec., **57**, Jan. to May.
11 Crelle's Journal, **168**, 4, **169**, 1 to 4.
12 Current Science, **1**, 6 to 12, **2**, 1.
13 Japanese Journal of Mathematics, **9**, 4.
14 Jahrbuch uber die Fortschritte der Mathematik, **42**, 5—8.
15 Jahresbericht der deutschen mathematiker Vereinigung, **42**, 9—12.
16 Japanese Journal of Mathematics, **9**, 3.
17 Journal of the University of Bombay, **1**, 2.
18 Mathematical Gazette, **16**, 221, **17**, 222, 223, 224 (3 copies each).
19 Mysore University Half Yearly Journal, **5**, 2, **6**, 1.
20 Mathematische Annalen, **107**, 4, 5, **108**, 1, 2, 3, 4.
21 Monthly Notices of the Royal Astronomical Society, **93**, 1 to 7.
22 Nieuw Archief voor Wiskunde, **17**.
23 Philosophical Magazine, **14**, 93 to 95, **15**, 96 to 102.
24 Philosophical Trans. of the Royal Society, **231**, Pp. 147—205; **232**, Pp. 1—283
25 Popular Astronomy, **40**, 9, 10, **41**, 1 to 6 (2 copies).
26 Proceedings of the Cambridge Philosophical Society, **28**, 4, **29**, 1, 2.
27 Proceedings of the Edinburgh Mathematical Society, **3**, 3.
28 Proceedings of the London Mathematical Society, **34**, 5, 6, 7, **35**, 2, 3, 4, 5. (2 copies).
29 Proceedings of the Physico-Mathematical Society of Japan, **14**, 9 to 11, **15**, 1, 2, 3, 4, 5.
30 Proceedings of the Royal Society of London, **138**, 836, **139**, 837—841, **140**, 842, **141**, 843.
31 Publications de la Faculte des Sciences, 153, 154, 156, 158.
32 Quarterly Journal of Mathematics, **3**, 12, **4**, 13, 14.
33 Rendiconti del Circolo Matematico di Palermo, **56**, 2, 3, 4, **57**, 1.
34 Revista Mathematica Hispano Americana, **7**, 9, 10.
35 Transactions of the Royal Society of South Africa, **21**, 1, 2.
36 Tohoku Mathematical Journal, **31**, 11.
37 Wiskundige Opgaven met de Oplossungen, Zestiende Deel Eerste Stuk.

### Books.

1 The distribution of Prime Numbers, Cambridge Tracts No. 30.
2 SIGLEY, D.T.: Groups involving a small number of complete sets of Conjugates.
3 VOLTERRA, V.: Flow of Electricity in a magnetic field.
4 WOO, K.: The projective transformation group of a hyperquadric in 4-dimensional space.
5 Madras University Calendar, 1932-33.
6 MACROBERT: Functions of a complex variable.

---

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

VOLUME I. NUMBER 2.

JUNE, 1933.

## CONTENTS

PAGES

Notes on the theory of Numbers: (1) Prime numbers:
By **S. Chowla** ... ... ... 41—48

Historical Notes in Felix Klein's Elementary Mathematics:
By **G. A. Miller** ... ... ... 49—51

On a pastime common among South Indian school children:
By **S. Sivasankaranarayana Pillai** ... ... 52—56

On the loci of the centres of equilateral triangles inscribed in or circumscribed about or self-polar to a given conic and of regular tetrahedra self-polar to a conicoid:
By **N. Durairajan** ... ... ... 57—62

Notes and Discussions:

On isogonal point-pairs whose pedal circle cuts the nine points circle at a constant angle:
By **A. Narasinga Rao**... ... ... 63—67

Gleanings ... ... ... ... 67—69

Announcements and News ... ... ... 68—70

Reviews ... ... ... ... 71

Solutions for Questions ... ... ... 72—78

Questions for Solutions ... ... ... 79—80

All communications intended for publication should be sent to the Editor, A. NARASINGA RAO, Annamalai University, Annamalainagar, South India. Communications relating to the receipt of the Journal, changes of address, and the purchase of current or back numbers of this journal should be sent to the Assistant Secretary, S. MAHADEVA IYER, Assistant Professor of Mathematics, Presidency College, Madras.

Remittances of membership fees and of subscriptions should be sent to the Treasurer, S. R. RANGANATHAN, Librarian, Madras University Library, Senate House, Madras.